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# Viscous dissipation effects on the asymptotic behaviour of laminar forced convection for Bingham plastics in circular ducts

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**Abstract**

The present study concentrates on the effects of viscous dissipation and the yield shear stress on the asymptotic behaviour of the laminar forced convection in a circular duct for a Bingham fluid. It is supposed that the physical properties are constant and the axial conduction is negligible. The asymptotic temperature profile and the asymptotic Nusselt number are determined for various axial distributions of wall heat flux which yield a thermally developed region. It is shown that if the asymptotic value of wall heat flux distribution is vanishes, the asymptotic value of the Nusselt number is zero. The case of the asymptotic wall heat flux distribution non-vanishing giving a value of the Nusselt number dependent on the Brinkman number and on the dimensionless radius of the plug flow region was also analysed. For an infinite asymptotic value of wall heat flux distributions, the asymptotic value of the Nusselt number depends on the dimensionless radius of the plug flow region and on the dimensionless parameter which depends on the asymptotic behaviour of the wall heat flux. The condition of uniform wall temperature and convection with an external isothermal fluid were also considered. The comparison with other existing solutions in the literature in the Newtonian case is analysed.



*Keywords:* Laminar forced convection; Viscous dissipation; Bingham plastic; Asymptotic behaviour; Variable wall heat flux; Analytical methods

## 1. Introduction

Laminar forced convection of a Bingham plastic in circular ducts is of great practical interest since one finds them in various industrial applications such as chemical food, polymers, cosmetics and pharmaceutical processing industries.

The laminar forced convection of Newtonian fluids following in circular ducts has been widely studied and the significant results obtained in this field are summarized by Shah and London [1]. The temperature field and the local Nusselt number in the thermal entrance region is evaluated by Quaresma and Cotta [2] with sinusoidal and exponential wall heat flux variation, in the case of negligible viscous dissipation and axial heat flux conduction in the fluid. The same work was carried out by Barletta and Zanchini [3] with power-law wall heat flux variation. In Ref. [4], a sufficient condition much broader than those previously found in the literature has been determined for the existence of fully thermally developed region. Under the same assumptions, Piva [5] proposed an analytical method based on confluent hypergeometric functions to predict the fully developed Nusselt number in the case of exponential wall heat flux.

The effect of viscous dissipation in the thermal entrance region was examined in several research works. One mentions those of Basu and Roy [6] for isothermal and constant heat flux boundary conditions. In the case

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**Nomenclature**

| | |
|---|---|
| $a$ | ratio of yield shear stress to wall shear stress |
| $Bi$ | Biot number, $h_e r_0/\lambda$ |
| $b(R)$ | solution of Eqs. (34) and (35) |
| $Br(X)$ | local Brinkman number, $(\mu_p u_m^2)/(2r_0)q_w(x)$ |
| $Br_\infty^{(s)}$ | singular value of $Br_\infty$ evaluated by Eq. (28) |
| $C$ | dimensionless constant employed in Eq. (33) |
| $c_p$ | specific heat at constant pressure |
| $f$ | function of $R$ employed in Eq. (14) or in Eq. (22) |
| $g$ | arbitrary function of $r$ and $x$ |
| $h_e$ | convection coefficient with a fluid external to the tube wall |
| $Nu$ | Nusselt number, $2r_0 q_w/[\lambda(T_w - T_b)]$ |
| $Pe$ | Peclet number, $2r_0 u_m \rho c_p/\lambda$ |
| $q_w$ | wall heat flux |
| $r$ | radial coordinate |
| $r_0$ | radius of the tube |
| $R$ | dimensionless radial coordinate, $r/r_0$ |
| $R'$ | dummy integration variable |
| $T$ | temperature |
| $T_0$ | inlet temperature distribution |
| $T_f$ | reference temperature of a fluid external to the tube wall |
| $u$ | velocity component in the axial direction |
| $u_m$ | mean axial velocity |
| $U$ | dimensionless axial velocity, $u/u_m$ |
| $x$ | axial coordinate |
| $X$ | dimensionless axial coordinate, $x/2r_0 Pe$ |

*Greek symbols*

| | |
|---|---|
| $\beta$ | dimensionless parameter defined in Eq. (30) |
| $\lambda$ | thermal conductivity of fluid |
| $\mu_p$ | plastic viscosity |
| $\omega$ | dimensionless parameter, $1 - (4a/3) + (a^4/3)$ |
| $\rho$ | fluid density |
| $\tau_c$ | yield shear stress |
| $\tau_w$ | wall shear stress |
| $\theta$ | dimensionless temperature, $\lambda(T - T_{0b})/\mu_p u_m^2$ |
| $\theta_f$ | dimensionless temperature, $\lambda(T_f - T_{0b})/\mu_p u_m^2$ |
| $\Theta$ | dimensionless temperature, $(T_w - T)/(T_w - T_b)$ |

*Subscripts*

| | |
|---|---|
| b | bulk quantity |
| w | wall condition |
| $\infty$ | quantity evaluated for $X \to +\infty$ |

of isothermal wall, the asymptotic value of the Nusselt number is $Nu_\infty = 48/5$, whereas in the case of constant wall heat flux, this value remains lower than 4.36. Zanchini [7] evaluated analytically the asymptotic temperature field and the asymptotic value of the Nusselt number for many wall heat flux distributions.

For non-Newtonian fluids, the results concerning the heat transfer by forced convection for power-law fluids in circular ducts are illustrated in Refs. [8,9]. For fully developed laminar flow, the effect of viscous dissipation is analysed by Barletta [10] for many axial distribution of wall heat flux which ensures the existence of a thermally developed region. Recently, Olek [11] obtained a general analytical solution for laminar forced convection in the thermal entrance region of a circular or parallel-plate duct with including axial heat conduction, convective boundary conditions and for fully developed velocity profile.

The problem of heat transfer for a Bingham plastic in laminar tube flow was studied by Johnston [12] by using a method based on the Sturm–Liouville transform theory. He concluded that the axial conduction can be neglected, if the Peclet number is higher than 1000. Vradis et al. [13] analysed numerically the heat transfer problem with viscous dissipation and without axial conduction, in the case of uniform wall temperature. Recently, Min et al. [14] studied analytically the fully developed and the thermally developing regimes of a Bingham plastic by employing the Frobenius method and the separation of variables by taking into account the viscous dissipation and the axial conduction with uniform wall temperature.

To our knowledge, no solution of the problem of forced convection with viscous dissipation in a circular duct of a Bingham plastic with no-uniform wall heat flux is available in the literature.

The aim of the present work is to apply the approach employed in Ref. [10] to the case of fully developed laminar forced convection in circular ducts for a Bingham plastic with viscous dissipation and negligible axial heat conduction in the fluid. Both the asymptotic Nusselt number and the asymptotic profile temperature are obtained for many axial distributions of wall heat flux which yield a thermally developed region. The effect of the dimensionless radius of the plug core and the Brinkman number are presented and compared with those obtained in previous works. The cases of constant wall temperature and convection with an external isothermal fluid were also considered.

## 2. Analysis

Let us consider a Bingham plastic of constant physical properties flowing in a circular duct of radius $r_0$, submitted to a variable axial wall heat flux $q_w(x)$. The flow is supposed to be steady, laminar, fully developed and axisymmetric.

The fully developed velocity profile for a laminar pipe flow of a Bingham plastic is given as follows [14,15]:

$$u(r) = \begin{cases} u_m \dfrac{2\left(1-\left(\dfrac{r}{r_0}\right)^2 - 2a\left(1-\dfrac{r}{r_0}\right)\right)}{1-\dfrac{4a}{3}+\dfrac{a^4}{3}} & \text{if } r_c \leqslant r \leqslant r_0 \\ u_m \dfrac{2(1-a)^2}{1-\dfrac{4a}{3}+\dfrac{a^4}{3}} & \text{if } 0 \leqslant r \leqslant r_c \end{cases} \tag{1}$$

where $a = \tau_c/\tau_w = r_c/r_0$ is the dimensionless radius of the plug flow region, $\tau_c$ the yield shear stress, $\tau_w$ the wall shear stress, $r$ the radial coordinate, $r_c$ the yield radius, and $u_m$ the mean value of velocity.

The equation energy and associated boundary conditions are given by

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{\lambda}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + \mu_p\left(\frac{du}{dr}\right)^2 - \tau_c\frac{du}{dr} \tag{2}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0, \quad \left.\frac{\partial T}{\partial r}\right|_{r=r_0} = \frac{q_w(x)}{\lambda} \tag{3}$$

$$T(r, x=0) = T_0(r) \tag{4}$$

where $\rho$, $\lambda$, $\mu_p$ and $c_p$ are the density of fluid, thermal conductivity, the plastic viscosity, and the specific heat at constant pressure, respectively.

The condition that leads to an asymptotic thermally developed region in the case of the forced convection problem described above, is defined by [16]

$$\lim_{x\to+\infty} \frac{T_w(x) - T(r,x)}{T_w(x) - T_b(x)} = \lim_{x\to+\infty} \Theta(r/r_0, x/2r_0Pe) = \Theta_\infty(r/r_0) \tag{5}$$

where $T_w(x)$ and $T_b(x)$ are the wall temperature and the bulk temperature, respectively, $Pe$ is the Peclet number, and $\Theta_\infty(r/r_0)$ is the asymptotic dimensionless temperature which is a continuous and differentiable function of $r$. The bulk value of an arbitrary function $g(r,x)$ is defined as

$$g_b(x) = \frac{2}{u_m r_0}\int_0^{r_0} g(r,x)u(r)r\,dr \tag{6}$$

If condition (5) is holds, the asymptotic value of the Nusselt number $Nu_\infty$ exists [16] and is given by

$$\lim_{x\to+\infty} Nu = 2r_0 \lim_{x\to+\infty} \frac{\left.\frac{\partial T}{\partial r}\right|_{r=r_0}}{T_w(x) - T_b(x)} = -2r_0 \left.\frac{d\Theta_\infty}{dr}\right|_{r=r_0} = Nu_\infty \tag{7}$$

By the same proof presented in Ref. [10], it is easy to check that the boundary value problem, expressed by Eqs. (2)–(4) has a unique solution, and both the asymptotic behaviour of the temperature field and of the Nusselt number are independent on the temperature distribution in the inlet section.

By introducing the dimensionless quantities

$$X = \frac{x}{2r_0Pe}, \quad R = \frac{r}{r_0}, \quad U(R) = \frac{u(r)}{u_m}, \quad \theta = \lambda\frac{T - T_{0b}}{\mu_p u_m^2} \tag{8}$$

Eqs. (2) and (3) can be rewritten in the dimensionless form

$$\frac{\partial}{\partial R}\left[R\frac{\partial \theta}{\partial R}\right] = \frac{RU}{4}\frac{\partial \theta}{\partial X} + \frac{4a}{1-4a/3+a^4/3}R\frac{dU}{dR} - R\left(\frac{dU}{dR}\right)^2 \tag{9}$$

$$\left.\frac{\partial \theta}{\partial R}\right|_{R=0} = 0, \quad \left.\frac{\partial \theta}{\partial R}\right|_{R=1} = \frac{1}{2Br(X)} \tag{10}$$

where $Br(X)$ is a local Brinkman number defined as

$$Br(X) = \frac{\mu_p u_m^2}{(2r_0)q_w(x)} \tag{11}$$

Integrating Eq. (9) over the interval $0 \leqslant R \leqslant 1$ and employing Eq. (10) yields

$$\frac{\partial \theta_b}{\partial X} = \frac{4}{Br(X)} + \frac{32}{1-4a/3+a^4/3} \tag{12}$$

where $\theta_b(X)$ is the bulk value of the dimensionless temperature $\theta(R,X)$.

## 3. Asymptotic behaviour of the temperature field

In this work, the asymptotic temperature field and the asymptotic Nusselt number are evaluated by considering three classes of functions $Br(X)$ which yield a thermally developed region.

*First case:* The axial distributions of wall heat flux are such that

$$\lim_{X\to+\infty} Br(X) = \pm\infty \tag{13}$$

Eq. (13) is satisfied for axial distributions of wall heat flux, which tend to zero when $X \to +\infty$. These distributions have a form of polynomial fractions with degree of the numerator lower than the degree of the denominator, or the exponentially decreasing functions. The solution of Eqs. (9) and (10) (separated solution due to the non-homogeneous boundary conditions), for large value of $X$, can be written as [10]

$$\theta(R,X) = \theta_b(X) + f(R) \tag{14}$$

where $f(R)$ is a continuous and differentiable function of $R$.

By substituting Eq. (14) in Eqs. (9) and (10), and by employing Eqs. (12) and (13), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R\frac{\mathrm{d}f}{\mathrm{d}R}\right]=\frac{8RU}{\omega}+\frac{4a}{\omega}R\frac{\mathrm{d}U}{\mathrm{d}R}-R\left(\frac{\mathrm{d}U}{\mathrm{d}R}\right)^2 \tag{15}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=0}=0,\quad \left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=1}=0 \tag{16}$$

where $\omega=1-(4a/3)+(a^4/3)$.

The integration of Eq. (15), by taking into account of the continuity of $f(R)$ in $R=a$ and of the vanishing bulk value of $f(R)$, and by employing Eqs. (5), (8) and (14), one obtains a following asymptotic temperature field

$$\Theta_\infty(R)=\frac{f(1)-f(R)}{f(1)} \tag{17}$$

$$\Theta_\infty(R)=\begin{cases}2-\dfrac{4R^2}{\omega}\left(1-\dfrac{R^2}{2}-2a\left(1-\dfrac{2R}{3}\right)\right)-\dfrac{2a^4}{3\omega} \\ \quad\text{if } a\leqslant R\leqslant 1\\ 2-\dfrac{4R^2(1-a)^2}{\omega}\\ \quad\text{if } 0\leqslant R\leqslant a\end{cases} \tag{18}$$

Eqs. (7) and (18) yield a vanishing asymptotic value of the Nusselt number

$$Nu_\infty=-2\left.\frac{\mathrm{d}\Theta_\infty}{\mathrm{d}R}\right|_{R=1}=0 \tag{19}$$

$$\lim_{X\to+\infty}\frac{1}{Br(X)}\frac{\mathrm{d}Br(X)}{\mathrm{d}X}=0 \tag{21}$$

where $Br_\infty$ is the asymptotic Brinkman number, and is a real number.

Conditions (20) and (21) are checked by uniform wall heat flux distributions, and when $q_w(X)$ is given by: polynomial functions, logarithmic functions, rational functions where the degree of the numerator is greater than or equal to the degree of the denominator, ... etc. Under these conditions, there exists an asymptotic fully developed region for the temperature field, and the solution of the Eqs. (9) and (10), for large values of $X$, is expressed by

$$\theta(R,X)=\theta_b(X)+\frac{f(R)}{Br(X)} \tag{22}$$

Substituting Eq. (22) in Eqs. (9) and (10) and taking into account Eqs. (12), (20) and (21) gives

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R\frac{\mathrm{d}f}{\mathrm{d}R}\right]=RU\left(1+\frac{8Br_\infty}{\omega}\right)+\left(\frac{4a}{\omega}R\frac{\mathrm{d}U}{\mathrm{d}R}-R\left(\frac{\mathrm{d}U}{\mathrm{d}R}\right)^2\right)Br_\infty \tag{23}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=0}=0,\quad \left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=1}=\frac{1}{2} \tag{24}$$

The integration of Eq. (23), by taking into account of the continuity of $f(R)$ in $R=a$ and of the vanishing bulk value of $f(R)$ [17] (for uniform inlet temperature), and on account of Eqs. (5), (8) and (22), the asymptotic temperature field $\Theta_\infty(R)$ can be written as

$$\Theta_\infty(R)=\frac{f(1)-f(R)}{f(1)}=\begin{cases}1-\left[\frac{1-2a}{2\omega}\left(1+\frac{8Br_\infty}{\omega}\right)R^2+\frac{4a}{9\omega}\left(1+\frac{12Br_\infty}{\omega}\right)R^3-\frac{1}{8\omega}\left(1+\frac{16Br_\infty}{\omega}\right)R^4+\frac{a^4}{6\omega}\ln(R)+C_1\right][f(1)]^{-1} & \text{if } a\leqslant R\leqslant 1\\ 1-\left[\frac{1}{2\omega}\left(1+\frac{8Br_\infty}{\omega}\right)\left((1-a)^2R^2-\frac{a^4}{6}\right)+\frac{a^4}{6\omega}\left(\ln(a)-\frac{7}{12}\right)+C_1\right][f(1)]^{-1} & \text{if } 0\leqslant R\leqslant a\end{cases} \tag{25}$$

Other authors found this result in the Newtonian fluid case [7] and in the power-law non-Newtonian fluid case [10].

Fig. 1 represents the evolution of the asymptotic temperature field $\Theta_\infty(R)$ for various values of the core radius $a$, when the Brinkman number tends to $\pm\infty$, for large values of $X$. It should be noted that the asymptotic temperature profile, which is given by the Eq. (18) increases with $a$.

*Second case:* The axial distributions of wall heat flux are such as

$$\lim_{X\to+\infty}Br(X)=Br_\infty \tag{20}$$

and, if $Br_\infty=0$

where

$$C_1=-\frac{a^4}{18\omega^2}\ln(a)-\frac{Br_\infty}{\omega^2}\left(1-\frac{4a}{3}-\frac{a^4}{3}\right)-\frac{1}{\omega^2}\left(\frac{7}{48}-\frac{257a}{630}+\frac{38a^2}{135}-\frac{a^4}{8}+\frac{5a^5}{27}+\frac{a^6}{15}-\frac{2a^7}{15}-\frac{13a^8}{1008}\right)$$

because $\omega=1-(4a/3)+(a^4/3)$

$$f(1)=\frac{1}{\omega^2}\left(\frac{11}{48}+\omega Br_\infty\right)-\frac{a^8}{18\omega^2}\ln(a)+\frac{a}{\omega^2}\left(-\frac{2}{45}+\frac{62a}{135}+\frac{a^3}{4}-\frac{10a^4}{27}-\frac{a^5}{15}+\frac{2a^6}{15}+\frac{13a^7}{1008}\right)$$

The Eqs. (7) and (25) yield the asymptotic value of the Nusselt number

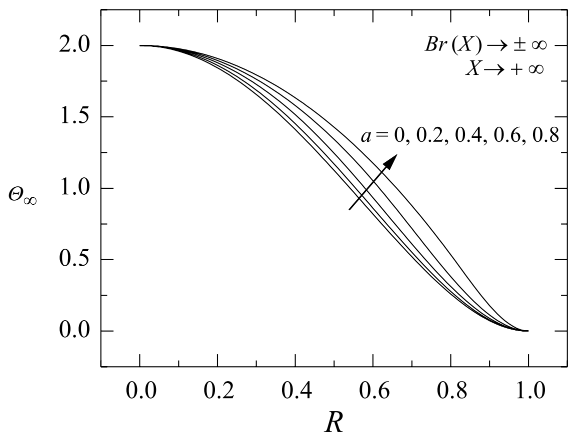


Fig. 1. Variation of fully developed temperature profile with respect to $a$.

$$Nu_\infty = -2\left.\frac{\mathrm{d}\Theta_\infty}{\mathrm{d}R}\right|_{R=1} = \frac{1}{f(1)} \tag{26}$$

Eq. (26) shows that the asymptotic value of the Nusselt number $Nu_\infty$ depends only on the core radius $a$ and on the asymptotic value $Br_\infty$ of the Brinkman number. When $a = 0$ (Newtonian flow case), the expression of the asymptotic value of $Nu$ coincides with that of Basu and Roy [6] and of Zanchini [7] in the case of uniform wall heat flux with viscous dissipation, such as

$$Nu_\infty = \frac{48}{11 + 48Br_\infty} \tag{27}$$

On the other hand, if $Br_\infty = 0$, Eq. (27) is reduced to the well known Newtonian value of $Nu_\infty = 48/11 = 4.3636$ in the case of uniform wall heat flux and negligible viscous dissipation [1].

Fig. 2 depict the asymptotic behaviour of the dimensionless temperature field for various values of the core radius $a$ and for $Br_\infty = -1$. This figure shows that there are values of $Br_\infty$ for which the asymptotic values of $Nu_\infty$ are negative. Indeed, the value of $Br_\infty$, which produces a singularity in the asymptotic value of $Nu$, is given by

$$Br_\infty^{(s)} = \frac{-\frac{11}{48}}{\omega} + \frac{a^8\ln(a)}{18\omega} - \frac{a}{\omega}\left(-\frac{2}{45} + \frac{62a}{135} + \frac{a^3}{4} - \frac{10a^4}{27} - \frac{a^5}{15} + \frac{2a^6}{15} + \frac{13a^7}{1008}\right) \tag{28}$$

Fig. 3 represents the evolution of $Br_\infty^{(s)}$ with respect to $a$, and shows that this value is always negative for any value of $a$. For $0 \leqslant a \leqslant 0.85$, one has $-12 \leqslant Br_\infty^{(s)} < 0$; and for $a \geqslant 0.85$, the value of $Br_\infty^{(s)}$ decreases suddenly down to a value of about $-120$. Table 1 presents the asymptotic values of the Nusselt number for various values of the core radius $a$, and shows that these values coincide with those reported by Zanchini [7] and by

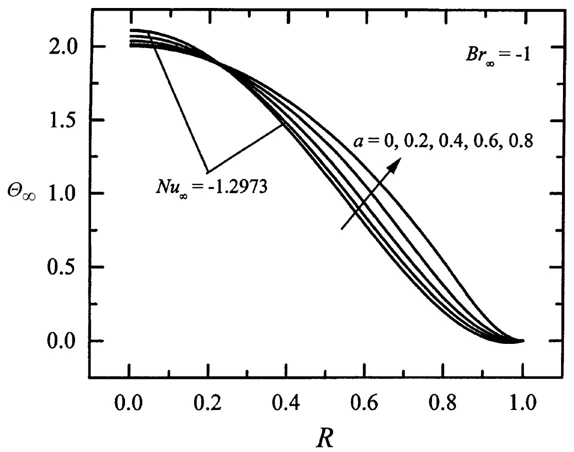


Fig. 2. Evolution the $\Theta_\infty(R)$ for various values of $a$ and $Br_\infty = -1$.

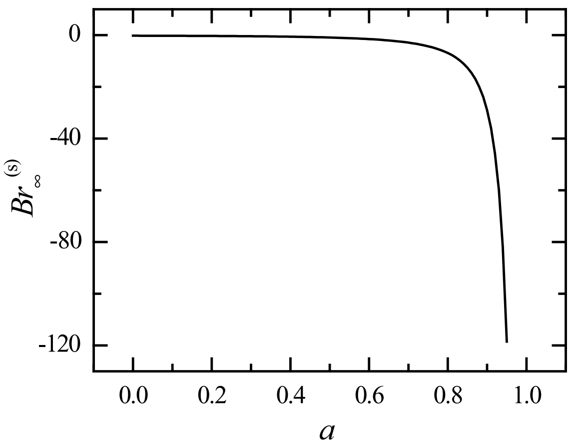


Fig. 3. Variation of $Br_\infty^{(s)}$ versus the core radius $a$.

Table 1
Values of $Nu_\infty$ evaluated by Eq. (26) and comparison with those of Barletta [10] and Zanchini [7] in the Newtonian case ($a = 0$)

| $a$ | $Br_\infty$ | | |
|---|---|---|---|
| | −1 | 0 | 1 |
| 0 | −1.2973 | 4.3636 | 0.8136 |
| 0.1 | −1.0767 | 4.4443 | 0.7253 |
| 0.2 | −0.8749 | 4.5528 | 0.6320 |
| 0.3 | −0.6914 | 4.6977 | 0.5342 |
| 0.4 | −0.5264 | 4.8883 | 0.4331 |
| 0.5 | −0.3804 | 5.1357 | 0.3313 |
| 0.6 | −0.2546 | 5.4543 | 0.2328 |
| 0.7 | −0.1505 | 5.8635 | 0.1431 |
| 0.8 | −0.0706 | 6.3911 | 0.0691 |
| 0.9 | −0.0187 | 7.0854 | 0.0187 |
| Zanchini [7] | −1.2973 | 4.3636 | 0.8136 |
| Barletta [10] | −1.2973 | 4.3636 | 0.8136 |

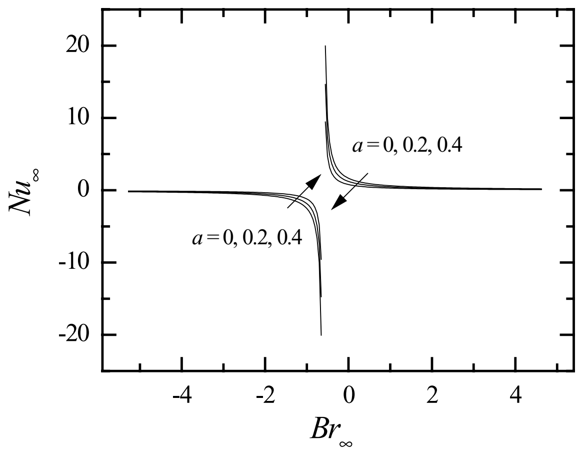


Fig. 4. Variation of $Nu_\infty$ versus $Br_\infty$ for different values of $a$.

Barletta [10] in the Newtonian case ($a = 0$). It is also noticed that when $Br_\infty = 0$ (negligible viscous dissipation), the value of $Nu_\infty$ is an increasing function of $a$. However, if $Br_\infty > Br_\infty^{(s)}$ the value of $Nu_\infty$ is a decreasing function of $a$ (Fig. 4), whereas for $Br_\infty < Br_\infty^{(s)}$ the asymptotic value of the Nusselt number is an increasing function of $a$ (Fig. 4).

*Third case:* The axial distributions of wall heat flux are such as

$$\lim_{X\to+\infty} Br(X) = 0 \tag{29}$$

and

$$\lim_{X\to+\infty} \frac{1}{Br(X)}\frac{\mathrm{d}Br(X)}{\mathrm{d}X} = -2\beta \tag{30}$$

where $\beta$ is a non-vanishing positive real number.

Eq. (29) shows that the effect of viscous dissipation is negligible in the thermally developed region. Eqs. (29) and (30) are satisfied by the axial wall heat flux distribution which tends to infinity when $X \to +\infty$ and which behave asymptotically as $Q(X)e^{2\beta X}$, where $Q(X)$ is a polynomial function, rational function where the degree of the numerator is greater than or equal to the degree of the denominator,...etc. Therefore, for each of these distributions the dimensionless temperature field for large value of $X$ can be expressed by Eq. (22).

By substituting Eq. (22) in Eqs. (9) and (10) and by taking into account of the Eqs. (12), (29) and (30), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R\frac{\mathrm{d}f}{\mathrm{d}R}\right] = \frac{RU}{2}(2+\beta f) \tag{31}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=0} = 0, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}R}\right|_{R=1} = \frac{1}{2} \tag{32}$$

Eq. (31) can be reduced to a first-order differential equation by employing the following transformation

$$f(R) = \frac{1}{\beta}\left[C\exp\int_0^R b(R')\,\mathrm{d}R' - 2\right] \tag{33}$$

where $C$ is a constant determined by the boundary condition at $R = 1$ and $b(R)$ is a continuous and differentiable function of $R$.

Substituting Eq. (33) into Eq. (31), gives

$$R\frac{\mathrm{d}b}{\mathrm{d}R} + b + Rb^2 = \frac{R\beta U}{2} \tag{34}$$

Eqs. (32) and (33) becomes then

$$b(0) = 0 \tag{35}$$

$$C = \frac{\beta}{2b(1)}\exp\left(-\int_0^1 b(R')\,\mathrm{d}R'\right) \tag{36}$$

Eq. (35) determines the boundary condition of Eq. (34) and Eq. (36) determines the constant $C$. Eq. (34) with its boundary condition (35) will be integrated numerically using the fourth-order Runge–Kutta method. The constant $C$ and the function $f(R)$ will be computed by the integration method of Simpson.

The function $\Theta_\infty(R)$ is evaluated by the following expression

$$\Theta_\infty(R) = \frac{f(1) - f(R)}{f(1)} \tag{37}$$

Taking into account Eqs. (7), (32), (33), (36) and (37), the asymptotic value of $Nu$ is expressed by

$$Nu_\infty = -2\left.\frac{\mathrm{d}\Theta_\infty}{\mathrm{d}R}\right|_{R=1} = \frac{2\beta b(1)}{\beta - 4b(1)} \tag{38}$$

The numerical method used requires about $10^6$ subdivisions of the interval $0 \leqslant R \leqslant 1$. The asymptotic values evaluated by the fourth-order Runge–Kutta method are compared with those of Shah and London [1], Piva [5] and Barletta [10] in the Newtonian fluid case ($a = 0$) with an exponentially varying wall heat flux (see Table 2). We note that the comparison between our theoretical results and those found in the literature in the Newtonian case is very satisfied. Fig. 5 represents the variation of $Nu_\infty$ versus $\beta$ for different values of $a$. This figure shows that for $\beta$ fixed, $Nu_\infty$ increases with $a$, and if $a$ is fixed, $Nu_\infty$ increases with $\beta$ (see Table 3).

The effect of the core radius $a$ on the evolution of the asymptotic temperature profile for different values of $\beta$ is presented in Fig. 6(a)–(c). One note that for large values of $\beta$ such as $\beta = 1000$ (Fig. 6(c)), the variation of $\Theta_\infty(R)$ does not vary significantly with respect to $a$, i.e. the effect of yield stress becomes negligible.

Table 2
Comparison between the asymptotic values of $Nu$ with those of the literature for $a = 0$ (Newtonian fluid)

| $\beta$ | Present work | Shah and London [1] | Piva [5] | Barletta [10] |
|---|---|---|---|---|
| 1 | 4.4475 | – | 4.4475 | 4.4475 |
| 5 | 4.7596 | 4.77 | 4.7596 | 4.7596 |
| 10 | 5.1066 | 5.11 | 5.1067 | 5.1067 |
| 20 | 5.6973 | 5.71 | 5.6974 | 5.6974 |
| 30 | 6.1925 | 6.21 | – | 6.1925 |
| 40 | 6.6221 | 6.64 | – | 6.6222 |
| 50 | 7.0040 | 7.02 | 7.0040 | 7.0040 |
| 60 | 7.3492 | – | – | 7.3493 |
| 70 | 7.6654 | – | – | 7.6655 |
| 80 | 7.9581 | – | – | 7.9582 |
| 90 | 8.2312 | – | – | 8.2313 |
| 100 | 8.4875 | – | 8.4877 | 8.4877 |
| 200 | 10.4844 | – | – | 10.4846 |
| 500 | 14.0950 | – | – | 14.0951 |
| 1000 | 17.7492 | – | 17.7495 | 17.7495 |
| 10 000 | 38.6601 | – | 38.6607 | 38.6607 |

## 4. Uniform wall temperature and convective boundary conditions

### 4.1. Constant wall temperature

If $T_w =$ constant, and by taking into account of the Eq. (8), the wall heat flux is expressed by

$$q_w = \frac{\lambda Nu}{2r_0}(T_w - T_b) = \frac{\mu_p u_m^2}{2r_0} Nu(\theta_w - \theta_b) \tag{39}$$

Eqs. (11), (12) and (39), for a fully developed temperature field $Nu = Nu_\infty$ yield

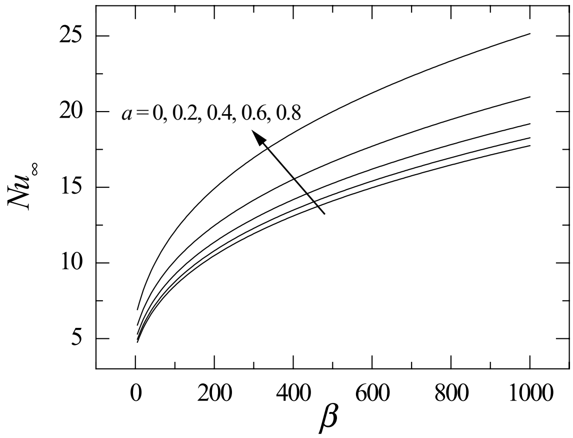


Fig. 5. Variation of $Nu_\infty$ versus $\beta$ for various values of $a$.

$$\theta_w - \theta_b = \theta_w \exp(-4Nu_\infty X) - \frac{8}{Nu_\infty \omega} \tag{40}$$

Substituting Eq. (40) into Eq. (39), $q_w$ is expressed in the fully developed region by

$$q_w = \frac{\mu_p u_m^2 Nu_\infty}{2r_0} \theta_w \exp(-4Nu_\infty X) - \frac{4\mu_p u_m^2}{r_0 \omega} \tag{41}$$

Taking into account Eq. (41), it is easily checked that the condition (20) is satisfied with the non-vanishing values of $Br_\infty$ such as

$$Br_\infty = -\frac{\omega}{8} \tag{42}$$

However, $\Theta_\infty(R)$ and $Nu_\infty$ are given, respectively, by Eqs. (25) and (26), with $Br_\infty$ expressed by the Eq. (42)

Table 3
Asymptotic values of $Nu$ for various values of $\beta$ and $a$

| $\beta$ | $a$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 1 | 4.528 | 4.637 | 4.782 | 4.974 | 4.970 | 5.546 | 5.961 | 6.502 | 7.211 |
| 5 | 4.841 | 4.951 | 5.098 | 5.293 | 5.551 | 5.888 | 6.331 | 6.917 | 7.704 |
| 10 | 5.189 | 5.301 | 5.451 | 5.652 | 5.919 | 6.273 | 6.746 | 7.384 | 8.264 |
| 20 | 5.784 | 5.900 | 6.057 | 6.268 | 6.552 | 6.936 | 7.461 | 8.192 | 9.238 |
| 30 | 6.283 | 6.404 | 6.568 | 6.789 | 7.088 | 7.498 | 8.067 | 8.877 | 10.072 |
| 40 | 6.716 | 6.843 | 7.013 | 7.243 | 7.556 | 7.988 | 8.596 | 9.476 | 10.807 |
| 50 | 7.102 | 7.233 | 7.409 | 7.648 | 7.974 | 8.426 | 9.068 | 10.010 | 11.465 |
| 60 | 7.451 | 7.586 | 7.768 | 8.015 | 8.353 | 8.823 | 9.496 | 10.494 | 12.064 |
| 70 | 7.770 | 7.901 | 8.097 | 8.351 | 8.700 | 9.188 | 9.889 | 10.939 | 12.616 |
| 80 | 8.066 | 8.209 | 8.402 | 8.664 | 9.023 | 9.526 | 10.253 | 11.351 | 13.128 |
| 90 | 8.342 | 8.489 | 8.687 | 8.955 | 9.324 | 9.842 | 10.593 | 11.735 | 13.607 |
| 100 | 8.601 | 8.752 | 8.954 | 9.221 | 9.608 | 10.139 | 10.913 | 12.096 | 14.058 |
| 200 | 10.621 | 10.801 | 11.042 | 11.370 | 11.822 | 12.462 | 13.409 | 14.911 | 17.584 |
| 500 | 14.274 | 14.510 | 14.826 | 15.254 | 15.845 | 16.686 | 17.943 | 19.998 | 23.966 |
| 1000 | 17.973 | 18.268 | 18.661 | 19.194 | 19.929 | 20.976 | 22.548 | 25.150 | 30.393 |
| 10 000 | 39.145 | 39.781 | 40.627 | 41.771 | 43.351 | 45.605 | 49.011 | 54.730 | 66.916 |

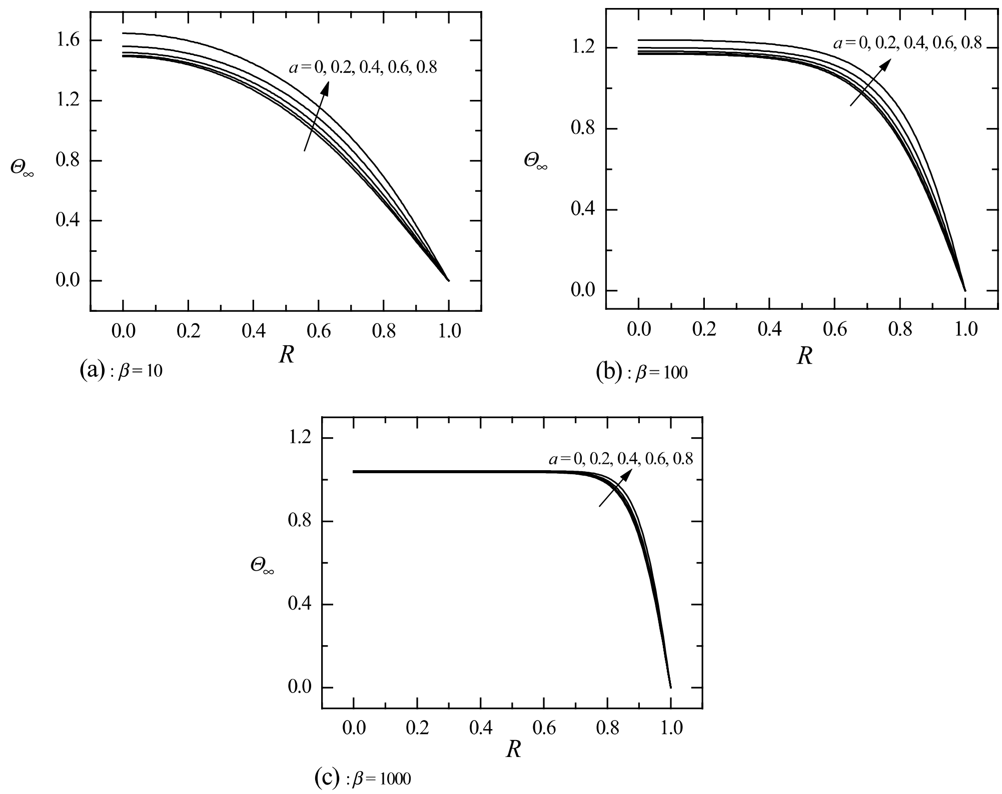


Fig. 6. Evolution the $\Theta_\infty(R)$ for various values of $\beta$: (a) $\beta = 10$, (b) $\beta = 100$, (c) $\beta = 1000$.

$$\Theta_\infty(R) = \begin{cases} 1 - \dfrac{\left[-\frac{2a}{9\omega}R^3 + \frac{1}{8\omega}R^4 + \frac{a^4}{6\omega}\ln(R) + C_1\right]}{f(1)} \\ \text{if } a \leqslant R \leqslant 1 \\ 1 - \dfrac{\left[\frac{a^4}{6\omega}\left[\ln(a) - \frac{7}{12}\right] + C_1\right]}{f(1)} \\ \text{if } 0 \leqslant R \leqslant a \end{cases} \tag{43}$$

$$Nu_\infty = -2\frac{d\Theta_\infty}{dR}\bigg|_{R=1} = \frac{1}{f(1)} \tag{44}$$

where

$$C_1 = -\frac{1}{\omega^2}\left[\frac{a^8\ln(a)}{18} + \frac{1}{48} - \frac{47a}{630} + \frac{8a^2}{135} - \frac{a^4}{8} + \frac{5a^5}{27} + \frac{a^6}{15} - \frac{2a^7}{15} + \frac{a^8}{1008}\right]$$

$$f(1) = \frac{1}{\omega^2}\left[-\frac{a^8\ln(a)}{18} + \frac{5}{48} + \frac{13a}{45} + \frac{32a^2}{135} + \frac{a^4}{6} - \frac{7a^5}{27} - \frac{a^6}{15} + \frac{2a^7}{15} - \frac{a^8}{1008}\right]$$

In the Newtonian fluid case ($a = 0$), one finds $Nu_\infty = 48/5$. This value coincides with the results reported in the literature [1,6,7,10]. Moreover, the asymptotic Nusselt number $Nu_\infty$ and the asymptotic temperature field $\Theta_\infty(R)$ depend only on the core radius $a$.

### 4.2. Convective boundary conditions

In this case, the duct wall exchanges the heat by convection with an external fluid having a uniform reference temperature, $T_f$, and a uniform convection coefficient $h_e$.

Introducing the Biot number, $Bi = h_e r_0/\lambda$, the wall heat flux is expressed as

$$q_w = \frac{\lambda Bi}{r_0}(T_f - T_w) = \frac{\mu_p u_m^2 Bi}{r_0}(\theta_f - \theta_w) \tag{45}$$

Taking into account Eqs. (11), (39) and (45), the Brinkman number is given by

$$Br(X) = \frac{Nu + 2Bi}{2NuBi(\theta_f - \theta_b)} \tag{46}$$

The substitution of Eq. (46) into Eq. (12), in the fully developed region, $Nu = Nu_\infty$, yields

$$\theta_f - \theta_b(X) = \theta_f \exp\left(-\frac{8Nu_\infty Bi}{Nu_\infty + 2Bi}X\right) - \frac{4(Nu_\infty + 2Bi)}{Nu_\infty Bi\omega} \tag{47}$$

By employing Eqs. (46) and (47), one obtains

$$Br(X) = \left[ \frac{2Nu_\infty Bi}{Nu_\infty + 2Bi} \theta_f \exp\left( -\frac{8Nu_\infty Bi}{Nu_\infty + 2Bi} X \right) - \frac{8}{\omega} \right]^{-1} \tag{48}$$

For large values of $X$, it is easy to check that $Br(X)$ given by Eq. (48) is reduced to that given by Eq. (42) as in the case $T_w$ = constant. Consequently, $\Theta_\infty(R)$ and $Nu_\infty$ will be expressed respectively by Eqs. (43) and (44) and do not depend on the value of $Bi$. This can be interpreted as convective boundary condition with $Bi$ tends to infinity. By taking into account of Eq. (11), the Eq. (48) ensures that, when $Bi \to \infty$ the wall heat flux is a finite value. Therefore, on account of Eq. (45), $T_w$ must tend to $T_f$. Fig. 7 describes the asymptotic behaviour of the temperature profile for various values of $a$. This figure shows

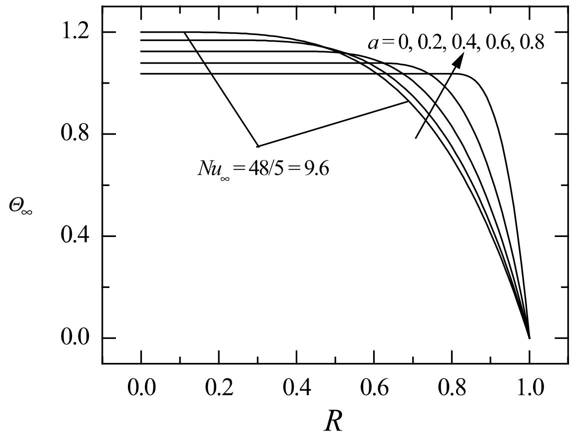


Fig. 7. Evolution the $\Theta_\infty(R)$ for various values of $a$ in the case of uniform wall temperature and of convective boundary conditions.

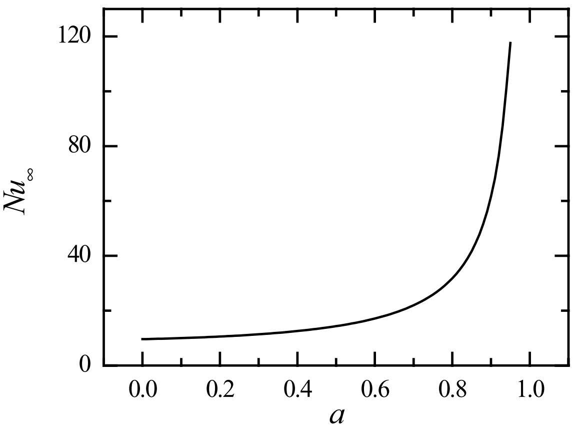


Fig. 8. Variation of $Nu_\infty$ versus $a$ in the case of uniform wall temperature and convective boundary conditions.

that the gradient of $\Theta_\infty(R)$ increases in the vicinity of the wall ($R = 1$) when the core radius $a$ increases. Fig. 8 shows that for $0 \leqslant a \leqslant 0.65$, the asymptotic value of the Nusselt number $Nu_\infty$ remains always lower than 20, whereas in the vicinity of $a \cong 8$, there is a significant increase in the value of $Nu_\infty$.

## 5. Conclusion

The fully developed laminar forced convection for a Bingham plastic in a circular duct is studied with three types of wall boundary conditions, namely: variable heat flux distributions, constant wall temperature and convection with an external isothermal fluid. The effects of viscous dissipation and the yield stress on the asymptotic temperature profile and the asymptotic Nusselt number are analysed.

For the axial wall heat flux distributions, three cases were considered. When the axial wall heat flux distributions $q_w(x)$ tends to zero for $x \to +\infty$, i.e. the local Brinkman number $Br(X)$ tends to infinity, (First case), the results show that the asymptotic value of the Nusselt number $Nu_\infty$ is zero. In the second case, where $q_w(x)$ does not tend to zero for $x \to +\infty$, while $(1/q_w(x))(\mathrm{d}q_w(x)/\mathrm{d}x)$ tend to zero, the asymptotic value of the Nusselt number is different from zero and depends only on the asymptotic Brinkman number $Br_\infty$ and of the core radius $a$. Moreover, viscous dissipation and the yield stress play a predominant role in the determination of the heat transfer characteristics of thermally fully developed flow. In addition, the value of $Br_\infty$ which produces a singularity, for each value of the core radius $a$, was presented. In the third case, $q_w(x)$ tends to infinity when $x \to +\infty$, while $(1/q_w(x))(\mathrm{d}q_w(x)/\mathrm{d}x)$ tends to a non-vanishing positive constant. It has been shown that the effect of viscous dissipation is negligible in the thermally developed region. The asymptotic temperature field and the asymptotic value of the Nusselt number are evaluated numerically for some values of $\beta$ and $a$.

For the boundary conditions of constant wall temperature and convection with an external isothermal fluid, one leads to same asymptotic values of the Nusselt number. In particular, it was shown that the asymptotic value of the Nusselt number in the case of convective boundary conditions is independent of the Biot number. The comparison between our theoretical results and those of the literature in the Newtonian fluid case is excellent.

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